

# An analytical formulation for roughness based on celular automata

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## Abstract

We present a method to derive the analytical expression of the roughness of a fractal surface whose dynamics is ruled by cellular automata. Starting from the automata, we write down the the time derivative of the height's average and variance. By assuming the equiprobability of the surface configurations and taking the limit of large substrates we find the roughness as a function of time. As expected, the function behaves as  $t^\beta$  when  $t \ll t_\times$  and saturate at  $w_s$  when  $t \gg t_\times$ . We apply the methodology to describe the etching model [1], however, the value of  $\beta$  we obtained are not the one predicted by the KPZ equation and observed in numerical experiments. That divergence may be due to the equiprobability assumption. We redefine the roughness with an exponent that compensate the nonuniform probability generated by the celular automata, resulting in an expression that perfectly matches the experimental results.

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## I. INTRODUCTION

The study of stochastic process has accelerated in the last decades, connected to disciplines such as economy, biology, meteorology and neuroscience. Much of the recent advances are due to the availability of fast and affordable computer clusters. Such development has became feasible, for instance, numerical simulation of large particle systems obeying simple repetitive rules which mimetize complex systems. Emergent information and properties have been obtained through several techniques and approaches.

The surface growth phenomena, when treated as a stochastic process, encompass a wide application field. Some examples of growth systems are corrosion [1, 2], fire propagation [3, 4], atomic deposition [5], evolution of bacterial colony [6, 7], and cellular automata models [8]. Models have been proposed and studied through experiments [4, 6, 7], analytical calculations [9], and computational simulations [1, 2].

In this work we obtain the roughness evolution analytically, for systems of dimension 1+1, referring to directions perpendicular and parallel to the substrate. Far from being mathematical idealizations, these systems have physical meaning. Phenomena with that dimensionality include bacterial colony growth on Petri dish [10–12], paper burning, ink diffusion on paper and turbulence of liquid crystals [13, 14]. In the case of paper, por example, the burned or stained frontier of the paper can be represented by  $h^f(x, t)$  where  $0 < x < L$  and  $L$  is the sheet width, being  $x$  and  $h^f$  measured along the directions mentioned above. The subscript  $f$  specify the reference frame fixed in a corner of the sheet.

Surfaces with different internal dynamics lead to distinct profiles, which can be characterized by different measures, the most important being the mean value and the standard deviation of the surface height. When related to surfaces, the standard deviation is often called roughness, defined as

$$w(L, t) = \sqrt{\frac{1}{L} \int_0^L [h^f(x, t) - \bar{h}(t)]^2 dx}. \quad (1)$$

Even if  $\bar{h}(t)$  increases continuously due to the growth process, the dynamic equilibrium lead to roughness saturation after a period of roughening buildup. The saturated roughness often is a function of the substrate size as the power law  $w_s \sim L^\alpha$ ,  $\alpha$  being the saturation exponent. The saturation occurs at a characteristic time ( $t_\times$ ), and follows the power law  $t_\times \sim L^z$ , where  $z$  is the dynamic exponent. Before saturation ( $t \ll t_\times$ ),  $w(L, t)$  evolves as a

power law with the growth exponent  $\beta$ ,  $w(L, t) \sim t^\beta$  [9]. These properties were incorporated in the Family-Vicsek scaling relation [15]

$$w(L, t) \propto L^\alpha f\left(\frac{t}{L^z}\right), \quad (2a)$$

where

$$f(x) \propto \begin{cases} x^\beta & \text{when } x \ll 1 \\ \text{const} & \text{when } x \gg 1 \end{cases}. \quad (2b)$$

Scaling techniques applied to the growth equations may be used to find the aforementioned exponents for certain universality classes and dimensions. Growth systems with the same exponents are considered to belong to the same universality class, connecting systems that seem unrelated. Some well known universality classes are the Edwards-Wilkinson and the KPZ [9].

## II. METHOD FOR OBTAINING THE ROUGHNESS EQUATION

In this section we present a method to obtain the equation describing the time evolution of the roughness. In the following section we will apply it to the etching model, notwithstanding, this method is quite general and can be extended to other models.

### A. The evolution of the mean squared roughness

The first step is to compute the change in the squared roughness, defined as  $w_q(t) \equiv w^2(t)$ , when the deposition occur in the site  $i$  of a substrate with roughness  $w$ . The squared roughness is equal to mean value of  $h_i^2(t)$ , with  $h_i$  defined at the reference frame of the mean height

$$h_i(t) = h_i^f(t) - \bar{h}(t), \quad (3)$$

where  $i$  is the particle position.

Our methodology assume that the change will depend only on the nearest neighbors of site  $i$ , therefore the variation of the squared roughness will be written as  $\Delta w_q(w, h_{i-1}, h_i, h_{i+1})$ . That assumption defines the celular automatas it can be applied to. We will do the average over the ensemble of all possible configurations with roughness  $w$ , defining  $p(w, h_{i-1}, h_i, h_{i+1})$

as the probability of the values  $h_{i-1}$ ,  $h_i$  and  $h_{i+1}$  for a given value of  $w$ . The evolution of the mean squared roughness is

$$\left\langle \frac{\Delta w_q}{\Delta t} \right\rangle = \int_{-\sqrt{Lw^2}}^{\sqrt{Lw^2}} \int_{-\sqrt{Lw^2-h_{i+1}^2}}^{\sqrt{Lw^2-h_{i+1}^2}} \int_{-\sqrt{Lw^2-h_{i+1}^2-h_i^2}}^{\sqrt{Lw^2-h_{i+1}^2-h_i^2}} \frac{\Delta w_q(w, h_{i-1}, h_i, h_{i+1})}{\Delta t} p(w, h_{i-1}, h_i, h_{i+1}) dh_{i-1} dh_i dh_{i+1}. \quad (4)$$

The integration limits encompass the configurations allowed by the definition of roughness, eq. (1). As will be discussed latter, the celular automata may prevent the occurrence of some configurations, which must have their probabilities assigned to zero.

### B. The rate of change of the quadratic roughness

Our approach involves calculating the increment of roughness when one iteration is performed. We use a discrete substrate with sites of length  $u = 1$  and evaluated the system before and after the deposition of one particle of height  $\Delta y \equiv 1$ . The squared roughness is affected by the following changes of  $h_i$

$$\begin{aligned} \text{Before: } h_i^2(t) &= \left[ h_i^f(t) - \bar{h}(t) \right]^2. \\ \text{After: } h_i^2(t + \Delta t) &= \left[ h_i^f(t + \Delta t) - \bar{h}(t + \Delta t) \right]^2 \\ &= \left[ h_i^f(t) + \Delta h_i(t) - \bar{h}(t) - \Delta \bar{h}(t) \right]^2 \\ &= h_i^2(t) + \varrho_i. \end{aligned} \quad (5)$$

with

$$\begin{aligned} \varrho_i &= 2h_i(t)\Delta h_i(t) - 2h_i(t)\Delta \bar{h}(t) \\ &\quad - 2\Delta h_i(t)\Delta \bar{h}(t) + [\Delta h_i(t)]^2 + [\Delta \bar{h}(t)]^2. \end{aligned} \quad (6)$$

Using the definition of roughness squared,  $w_q(t) \equiv w^2(t)$ , we write

$$\begin{aligned} \text{Before: } w_q(t) &= \frac{1}{L} \sum_i h_i^2(t). \\ \text{After: } w_q(t + \Delta t) &= \frac{1}{L} \sum_i [h_i^2(t) + \varrho_i]. \end{aligned} \quad (7)$$

The unity of time we use corresponds to  $L$  iterations, therefore,  $\Delta t = 1/L$  and the rate of change of  $w_q(t)$  is

$$\frac{\Delta w_q}{\Delta t} = -L[\Delta \bar{h}(t)]^2 + \frac{1}{L} \sum_i \{2h_i(t)\Delta h_i(t) + [\Delta h_i(t)]^2\}, \quad (8)$$

where we have used  $\sum_i h_i(t) = 0$  and  $\sum_i \Delta h_i(t) = L\Delta \bar{h}(t)$ .

Eq. (8) is a general formula for the increment of quadratic roughness, independent of the iterative algorithm. In order to obtain the roughness for a specific algorithm, it is necessary to know the values of  $\Delta h_i(t)$  and  $\Delta \bar{h}(t)$ .

On these grounds, each model results in different values for  $\Delta w_q(w, h_{i-1}, h_i, h_{i+1})/\Delta t$  and  $p(w, h_{i-1}, h_i, h_{i+1})$ , which must be deducted for each case. In the next two subsections we will assume the equiprobability of the accessible configurations to eliminate the dependence on  $p(w, h_{i-1}, h_i, h_{i+1})$ .

### C. The equiprobability of the configurations

Before proposing an expression  $p(w, h_{i-1}, h_i, h_{i+1})$  we must remember that there is a finite number of possible substrate configuration for each value of  $w$ . This finite number is the result of the restrictions of  $h_i$  imposed by eqs. (1) and (3):

$$h_1^2 + h_2^2 + \dots + h_L^2 = Lw^2 \quad (9a)$$

$$h_1 + h_2 + \dots + h_L = 0 \quad (9b)$$

Eqs. (9) define a hyperplane and the surface of a hypersphere of radius  $w\sqrt{L}$  both in  $L$ -D, i.e., in a space of  $L$  dimensions. From the intersection of these two subspaces, a spheric *surface* results, which is  $(L-2)$ -D. For each combination of  $h_{i-1}$ ,  $h_i$ , and  $h_{i+1}$ , the remaining  $L-3$   $h$ 's form another spheric *surface*, now with dimension  $L-5$ . The *area* of these spheric surfaces, with  $L-2$  and  $L-5$  dimensions, will be called, respectively,  $A_T$  and  $A_p$ .

While all possible surface configurations of a given value of  $w$  belong to the  $(L-2)$ -D surface, the subset of them for which the values of the triad  $h_{i-1}$ ,  $h_i$ , and  $h_{i+1}$  are known belongs to the  $(L-5)$ -D surface. We will assume the equiprobability of the configurations allowed by eqs. (9), consequently, the probability of a given triad is

$$p(w, h_{i-1}, h_i, h_{i+1}) = \frac{A_p}{A_T}. \quad (10)$$

The assumption of the equiprobability of the configurations, used when deriving that equation, disregard two important properties of the dynamics, which is not harmless. The first is the inaccessibility, by the celular automata, of several configurations allowed by eqs. (9). The second is the different probability of each configuration generated by the celular automata.

Eq. (10) can be rewritten by substituting the expression for the area of a hypersphere,

$$p(w, h_{i-1}, h_i, h_{i+1}) = \frac{S_{L-5} R_p^{L-5}}{S_{L-2} R_T^{L-2}}, \quad (11)$$

where  $S_D$  are constants with depend on the dimension  $D$ , and  $R_p$  and  $R_T$  are the radius of the corresponding hyperspheres. To make the notation less clumsy, we will assume, in the remaining of this subsection, a deposition at  $i = 2$ , implying that only the sites  $i = 1, 2$  ou  $3$  may be affected.

The plane of eq. (9b) contains the center of the sphere of eq. (9a), therefore, the sphere defined by their intersection also has radius  $R_T = w\sqrt{L}$ . The value of  $R_p$  may be obtained by rewriting eqs. (9) as

$$h_4^2 + h_5^2 + \dots + h_L^2 = Lw^2 - h_1^2 - h_2^2 - h_3^2 \quad (12a)$$

$$h_4 + h_5 + \dots + h_L = -(h_1 + h_2 + h_3) \quad (12b)$$

i.e., a hypersphere with superficial area  $A_p$  of dimension  $L - 5$ . These two equation may be combined as

$$-2 \sum_{\substack{i,j=4 \\ i \neq j}}^L h_i h_j = Lw^2 - h_1^2 - h_2^2 - h_3^2 - (h_1 + h_2 + h_3)^2, \quad (13)$$

which we rewrite in a matricial form:

$$\begin{vmatrix} h_4 & h_5 & \dots & h_L \end{vmatrix} \begin{vmatrix} 0 & -1 & \dots & -1 \\ -1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \dots & -1 & 0 \end{vmatrix} \begin{vmatrix} h_4 \\ h_5 \\ \vdots \\ h_L \end{vmatrix} = Lw^2 - h_1^2 - h_2^2 - h_3^2 - (h_1 + h_2 + h_3)^2. \quad (14)$$

The eigenvalues of the above square matrix are  $\lambda_4 = -(L - 4)$ ,  $\lambda_5 = 1$ ,  $\lambda_6 = 1$ ,  $\dots$ ,  $\lambda_L = 1$ . From these eigenvalues we can define a linear transformation to the set of variables  $h'_4, \dots, h'_L$  that eliminated the crossed terms,

$$-(L - 4)h_4'^2 + h_5'^2 + \dots + h_L'^2 = Lw^2 - h_1^2 - h_2^2 - h_3^2 - (h_1 + h_2 + h_3)^2. \quad (15)$$

Transformed variable  $h'_4$  is equal to the left hand side of eq. (12b)

$$\begin{aligned} h'_4 &= \frac{1}{\sqrt{L-3}}(h_4 + h_5 + \dots + h_L) \\ &= -\frac{1}{\sqrt{L-3}}(h_1 + h_2 + h_3). \end{aligned} \quad (16)$$

With this transformation, eq. (14) becomes

$$h'^2_5 + \dots + h'^2_L = Lw^2 - h_1^2 - h_2^2 - h_3^2 - \frac{(h_1 + h_2 + h_3)^2}{L-3}. \quad (17)$$

The radius of this hypersphere is given by

$$R_p = \left[ Lw^2 - h_1^2 - h_2^2 - h_3^2 - \frac{(h_1 + h_2 + h_3)^2}{L-3} \right]^{\frac{1}{2}}. \quad (18)$$

We can now rewrite eq. (11) with the values  $R_T$  and  $R_p$  in the asymptotic case  $L \rightarrow \infty$ ,

$$p(w, h_1, h_2, h_3) = \eta(L) \frac{[Lw^2 - h_1^2 - h_2^2 - h_3^2]^{\frac{L-5}{2}}}{(Lw^2)^{\frac{L-2}{2}}}, \quad (19)$$

with  $\eta(L) = S_{L-5}/S_{L-2}$ .

#### D. The roughness evolution with the equiprobability assumption

A more convenient expression for roughness squared, eq. (4), is possible by doing the change of coordinates

$$\begin{aligned} h_{i-1} &= \sqrt{L}w \sin \rho \cos \theta \\ h_i &= \sqrt{L}w \sin \rho \sin \theta \cos \varphi \\ h_{i+1} &= \sqrt{L}w \sin \rho \sin \theta \sin \varphi \end{aligned} \quad (20)$$

with the variables defined in the intervals

$$0 \leq \rho \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi. \quad (21)$$

The probability (19) expressed in these coordinates is

$$p(w, \rho, \theta, \varphi) = \eta(L) (Lw^2)^{-\frac{3}{2}} \cos^{L-5} \rho. \quad (22)$$

We can now rewrite the evolution of the squared roughness averaged over the ensembles, eq. (4):

$$\left\langle \frac{\Delta w_q}{\Delta t} \right\rangle = \eta(L) \int_0^{2\pi} \int_0^\pi \int_0^{\frac{\pi}{2}} \frac{\Delta w_q(w, \rho, \theta, \varphi)}{\Delta t} \sin^2 \rho \cos^{L-4} \rho \sin \theta d\rho d\theta d\varphi \quad (23)$$

In the above expression we employed the jacobian

$$dh_{i-1} dh_i dh_{i+1} = (Lw^2)^{\frac{3}{2}} \sin^2 \rho \cos \rho \sin \theta d\rho d\theta d\varphi. \quad (24)$$

While eq. (4) is exact, eq. (23) is an approximation based in the equiprobability assumption. Since we eliminated the dependence on  $p(w, h_{i-1}, h_i, h_{i+1})$ , the left hand side of that simplified equation depends on the roughness only through the term

$$\frac{\Delta w_q(w, \rho, \theta, \varphi)}{\Delta t}, \quad (25)$$

facilitating the resolution of the roughness equation. The celular automata used in each model will determine the value of this term, calculate through eq. (8).

### III. EXAMPLE: ETCHING MODEL

We will now calculate the evolution of the mean value of  $w_q$ , eq. (23), for the etching model. The etching model proposed in 2001 by Mello, Chaves, and de Oliveira [1] belongs to the *KPZ* universality class [9]. The model mimics the corrosion of a crystal surface by a solvent. A time evolution  $\Delta t$  is defined as one iteration of the following celular automata:

1. Randomly choose a site  $i \in [1..L]$ .
2. If  $h_{i-1}^f(t) < h_i^f(t)$  do  $h_{i-1}^f(t + \Delta t) = h_i^f(t)$ .
3. If  $h_{i+1}^f(t) < h_i^f(t)$  do  $h_{i+1}^f(t + \Delta t) = h_i^f(t)$ .
4. Do  $h_i^f(t + \Delta t) = h_i^f(t) + 1$ .

The algorithm implements a cell removal probability that is proportional to the number of the exposed faces of the cell, a reasonable approximation of the etching process. It could also describe a deposition where each exposed face has the same attachment probability. For that reason, it can be referred either as particle removal or deposition.

The scaling exponents found by Mello *et al.* in  $1 + 1$  dimension were  $\alpha = 0.491$  and  $\beta = 0.330$ , placing the model within the *KPZ* universality class ( $\alpha = 1/2$ ,  $\beta = 1/3$ ) [1]. Other studies analyzed the model in  $1 + 1$  and  $2 + 1$  dimensions, focusing on aspects such



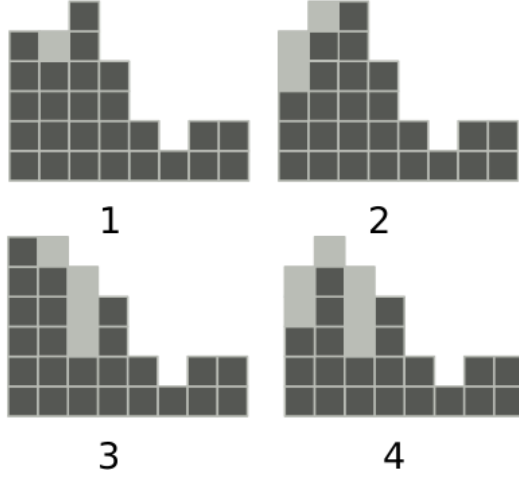


FIG. 1: Effects of deposition of a cell in site 2, classified in four cases.

as dynamic behavior of the roughness and comparisons with other models belonging to the KPZ universality class [2, 16–20].

Before applying eq. (8) to the etching model, it is necessary to know the values of  $\Delta h_i(t)$  and  $\Delta \bar{h}(t)$  for one iteration. While the first may be directly obtained from the model rules, the second must be separated in four possibilities, depending on the neighbors of the deposition site  $i$ . These four situations are shown in Fig. (1) together with the added cells when site  $i$  is selected. The effect of them on  $\Delta \bar{h}(t)$  are

$$\Delta \bar{h}(t) = \begin{cases} \text{Case 1: } \frac{1}{L} \Delta y. \\ \text{Case 2: } \frac{1}{L} [\Delta y + (h_i - h_{i-1})]. \\ \text{Case 3: } \frac{1}{L} [\Delta y + (h_i - h_{i+1})]. \\ \text{Case 4: } \frac{1}{L} [\Delta y + (h_i - h_{i-1}) + (h_i - h_{i+1})]. \end{cases} \quad (26)$$

The rate of change of  $w_q$ , eq. (8), can be separated in four parts corresponding to the cases of eq. (26). We represent each of them as  $\left(\frac{\Delta w_q}{\Delta t}\right)_j$ , with  $j = [1..4]$ . From eq. (20) we conclude that  $h_i \sim w$ , so, eq. (8) implies that  $\left(\frac{\Delta w_q}{\Delta t}\right)_j$  is a quadratic function of  $w$ . Since the integral (23) is a function of  $w$  only through  $\Delta w_q/\Delta t$ , all dependence on  $w$  is within that term, resulting in a quadratic equation for the roughness dynamics,

$$\left\langle \frac{\Delta w_q}{\Delta t} \right\rangle = -c_a w^2 - c_b w - c_c, \quad (27)$$

where the minus signals were include to simplify forthcoming calculations. In the limit

$\Delta t = 1/L \rightarrow 0$  we can do

$$\left\langle \frac{\Delta w_q}{\Delta t} \right\rangle \rightarrow \frac{dw_q}{dt} = 2w \frac{dw}{dt}. \quad (28)$$

By replacing the Eq. (27) in this expression, we obtain

$$-c_a dt = \frac{2w dw}{w^2 + \frac{c_b}{c_a}w + \frac{c_c}{c_a}} \equiv \frac{2w dw}{(w - w_1)(w - w_2)} \equiv \frac{A dw}{w - w_1} + \frac{B dw}{w - w_2}, \quad (29)$$

where  $w_1$  and  $w_2$  are the roots of  $w^2 + \frac{c_b}{c_a}w + \frac{c_c}{c_a} = 0$ , and  $A = \frac{2w_1}{w_1 - w_2}$ ,  $B = \frac{-2w_2}{w_1 - w_2}$ . The solution of the above differential equation is

$$\left( \frac{w - w_1}{w_0 - w_1} \right)^A \left( \frac{w - w_2}{w_0 - w_2} \right)^B = e^{-c_a t}, \quad (30)$$

being  $w_0$  the initial roughness.

We have found an implicit equation for the roughness as a function of time. Knowing that that equation and the Family-Vicsek scaling have both the characteristic time  $t_\times$ , we conclude that  $t_\times = 1/c_a$ . We will show briefly that  $w_1 w_2 < 0$ , and, if we choose  $w_1 > 0$ , then  $w_2 < 0$ ,  $A > 0$  and  $B > 0$ . The limit  $w \rightarrow w_s$  when  $t \rightarrow \infty$  establishes  $w_1 = w_s$ . Finally, roughness  $w$  depends on  $L$  through the constants  $t_\times$  and  $w_s$ , therefore, if we want the remaining of the equation to be independent of  $L$ , we must have  $w_2 = -\lambda w_s$  being  $\lambda$  a positive proportionality factor independent of  $L$ . Incorporating that reasoning in eq. (30), and considering a flat initial substrate, we can write

$$\left( 1 - \frac{w}{w_s} \right)^{\frac{2}{1+\lambda}} \left( 1 + \frac{1}{\lambda} \frac{w}{w_s} \right)^{\frac{2\lambda}{1+\lambda}} = e^{-t/t_\times}. \quad (31)$$

The initial growth, i.e., when  $w \ll w_s$ , is controlled by the exponent  $\beta$ . In order to make explicit that dependence, we expand the left hand side of Eq. (31) up to the second order, which result in

$$1 + \frac{1}{\lambda} \frac{w^2}{w_s^2} + O(3) = e^{-t/t_\times}. \quad (32)$$

The inversion of that equation is

$$w(t) \approx w_s \sqrt{\lambda} (1 - e^{-t/t_\times})^{1/2}. \quad (33)$$

This expression for  $w$  is real only if  $\lambda > 0$ , implying  $w_1 w_2 < 0$ , as we said. From the limit of the above expression,

$$\lim_{t \rightarrow 0} w(t) = w_s \sqrt{\lambda} \left( \frac{t}{t_\times} \right)^{1/2}, \quad (34)$$

we find  $\beta = 1/2$ .

It is interesting to make it clear that if  $\lambda = 1$ , Eq. (33) is identical to Eq. (31), not an approximation of it.

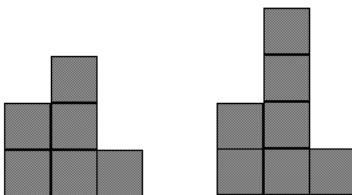


FIG. 2: Two possible configurations that can not be generated by the etching model.

### A. The aftermath of equiprobability assumption

The exponent  $\beta = 1/2$  found in the previous section is not the one expected for the KPZ universality class, which we know the etching model belongs to. The main approximation to be blamed for that disagreement is the equiprobability assumption, explicitly, that all configurations not forbidden by Eqs. (9) are allowed, and that they have all the same probability.

In growth models based in cellular automata, the internal rules of the model forbid certain configurations. For instance, the algorithm that governs the etching model can never lead to the configurations shown in the Fig. 2. According to the model, when one particle is deposited on the top of the second site, the first and third sites grow up to the earlier height of the second site, which is not the case of Fig. 2. It results that these are prohibited configurations.

Beside that, there is no reason to suppose that the dynamics results in equal probability for the allowed configurations. Indeed, numerical experiments with the etching model demonstrated that  $p(w, h_{i-1}, h_i, h_{i+1})$  doesn't agree with eq. (19).

The equiprobability assumption disregard the ban of some configurations and the non-uniformity of the probability distribution, allowing us to express the probability  $p(w, h_{i-1}, h_i, h_{i+1})$  as the ratio of the area of the partial hypersphere (defined by  $w, h_{i-1}, h_i, h_{i+1}$ ) and the area of the total hypersphere (defined by  $w$ ). However, the resulting expression for  $w$ , Eq. 33, has not the expected exponent  $\beta$ .

Aiming to circumvent the effect of the equiprobability approximations, we write the *real roughness of the etching model*,  $w^e$ , as a function of the roughness  $w$ , which does not include those features,

$$w = w_s \left( \frac{w^e}{w_s^e} \right)^\nu, \quad (35)$$

where the superscript refers to saturation. There are good reasons to reduce all the possibilities to this form. First,  $w$  must increase monotonously with  $w^e$ , they must agree for null roughness, and they have their maximum at saturation. This is the simplest form that that fulfil those requirements. Second, for a hypersphere of Euclidean dimension  $d$ , and radius  $\sim R$ , the surface area grows as  $R^{d-1}$ , while surfaces such as those originated by etching, or domains as in phase transition are expected to grow as  $S_f \propto R^{d_f-1}$ , where  $d_f$  is the fractal dimension. I.e., they must have a fractal character and dimension  $d_f$ . In this way, Eq. (35) is not only the simplest form, it is the only possibility.

If we substitute eq. (35) in eq. (33) we can write

$$w^e(t) = w_s^e \left[ 1 - \exp \left( -\frac{t}{t_\times} \right) \right]^\beta. \quad (36)$$

where  $\beta = \frac{1}{2\nu}$ . If we explicitly make  $w_s^e \propto L^\alpha$  and  $t_\times \propto L^z$ , that expression becomes one possible form of Family-Vicsek scaling relation, Eq. (2).

Although that equation is an expansion strictly valid only for  $t \ll t_\times$  it nicely fit to the results of the numerical simulations in the whole range of  $t$ , as can be seen in Fig. (3). Each curves of that figure was obtained by fitting its parameters  $\beta$ ,  $t_\times$ , and  $w_s$  to the corresponding data points. Data from simulation of other surface evolution models (RSOS, Edwards-Wilkinson) and higher dimensions have shown the same exceptional agreement with Eq. (36).

#### IV. CONCLUSIONS

In this work, we presented a method to obtain the roughness equation of the models. The method is based on the ratio of the hypersphere areas, interpreted as the probability of occurrence of determined configurations of the interface. The hyperspheres method has potential of application in automata cellular models which depend only on the nearest neighbors but it needs to be built differently for each type of algorithm. The algorithms need to act only in the nearest neighbors and the systems need to be one-dimensional. If approximations similar to equiprobability assumption are necessary to solve these models, transformation similar to Eq. (35) may be necessary.

The equation possesses three parameters, each of them is connected with one of the growth exponents. The modified equation is relevant not only because it fits well to the

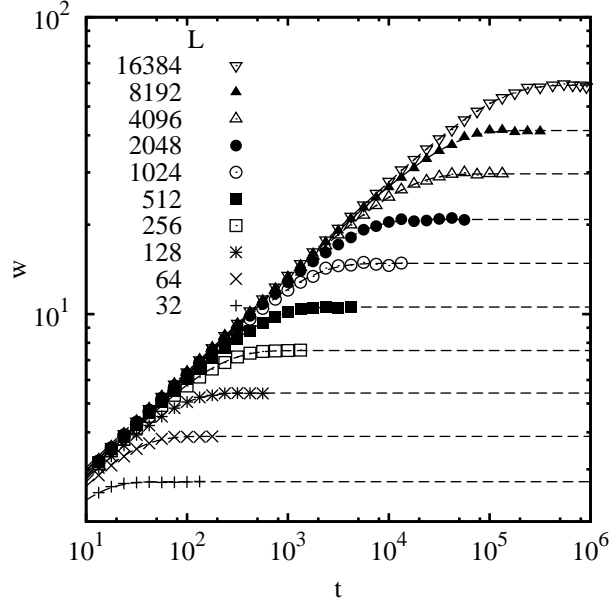


FIG. 3: Roughness as a function of the time. The points are the results of numerical simulation of etching model for several substrat lengths. The curves are fittings of Eq. (36) to the data points.

data, but also because we can also be used as fitting function for obtaining the value of the main parameters of the model,  $w_s^e$ ,  $\beta$ , and  $t_\times$ . In correlated stochastic phenomena , analytical results are rather difficult to obtain. In this way we hope that this work may inspire research into those systems where even not exact solutions can be considered major results [21].

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